2017

MATHEMATICS

(Major)

Paper : 3.1

(Abstract Algebra)

Full Marks : 80

Time : 3 hours

The figures in the margin indicate full marks for the questions

1. Answer the following as directed : $1 \times 10 = 10$

(a) A surjective homomorphism from a group to another group is called

(i) endomorphism

(ii) automorphism

(iii) monomorphism

(iv) epimorphism

(Choose the correct option)

(b) State whether the following statement is true or false :

"Every integral domain is a field."

(c) State whether the following statement is true or false :

The ring of all 2×2 matrices over reals under matrix addition and multiplication is an integral domain.

(d) If the rings R and S are of characteristics m and n respectively, then the characteristics of the product ring $R \times S$ is

(i) mn

(ii) m^n

(iii) $\text{lcm}[m, n]$

(iv) $\text{gcd}[m, n]$

(Choose the correct option)

(e) Define inner automorphism of a group G.

(f) If G is a non-Abelian group of order p^3, where p is a prime, then order of $Z(G)$ (the centre of G) is

(i) either 1 or p

(ii) either p or p^2

(iii) either p^2 or p^3

(iv) either 1 or p^3

(Choose the correct option)

(g) State whether the following statement is true or false :

"Abelian group of order 15 is always cyclic."

(h) Define kernel of a ring homomorphism.

(i) Define subring of a ring.

(j) Give the reason why the ideal $\langle 6 \rangle = \{6n : n \in Z\}$ is not a prime ideal of the ring of integers Z.

2. Answer the following : $2 \times 5 = 10$

(a) Prove that a group G is Abelian if the map $\mu : G \to G$ defined by $\mu(x) = x^{-1}$, $\forall x \in G$ is a homomorphism.

(b) Consider the homomorphism ψ from the group G to the group G'. Show that if G is simple, then either ψ is one-to-one or ψ maps each element of G to the identity element of G'.

(c) If L is a left ideal of a ring R, then show that $\lambda(L) = \{x \in R : xa = 0 \ \forall a \in L\}$ is an ideal of R.

(d) State Sylow's first and second theorems.

(e) Give example (with justification) to show that quotient ring of an integral domain may not be an integral domain.

3. Answer any *four* of the following : 5×4=20

(a) Show that the relation of isomorphism in the set of all groups is an equivalence relation.

(b) Show that every non-zero finite integral domain is a field.

(c) For any group G, prove that

$$\frac{G}{Z(G)} \cong I(G)$$

Here, $Z(G)$ is the centre of G and $I(G)$ is the inner automorphism group of G.

(d) If R is a commutative ring, then show that an ideal P of R is prime if and only if for any two ideals A and B of R,

$$AB \subseteq P \Rightarrow \text{either } A \subseteq P \text{ or } B \subseteq P$$

(e) Prove that a non-empty subset W of a vector space $V(F)$ is a subspace of V if and only if $\alpha u + \beta v \in W$, $\forall \alpha, \beta \in F$ and $\forall u, v \in W$.

(f) Show that $\langle 4 \rangle = \{4n : n \in Z\}$ is a maximal ideal of the ring of even integers $(E, +, .)$. Is $\langle 4 \rangle$ a prime ideal of E? 4+1=5

4. Answer the following questions : 10×4=40

(a) Let G be any group. If H is any subgroup and N be any normal subgroup of G, then show that—

(i) $H \cap N$ is a normal subgroup of G;

(ii) N is normal in $HN = \{x = hn : h \in H, n \in N\}$;

(iii) $\frac{HN}{N} \cong \frac{H}{H \cap N}$. 2+2+6=10

Or

Let f be a homomorphism from the group G onto the group G' and H be a subgroup of G, H' a subgroup of G'. Show that—

(i) $f(H)$ is a subgroup of G';

(ii) $f^{-1}(H')$ is a subgroup of G containing $\ker f$, where $f^{-1}(H') = \{x \in G : f(x) \in H'\}$;

(iii) there exists one-to-one correspondence between the sets of subgroups of G containing $\ker f$ and subgroups of G'. 2+3+5=10 *(b)* Define ideal of a ring. If A and B are two ideals of a ring R, then show that their sum $A+B=\{a+b : a \in A, b \in B\}$ is also an ideal of R containing both A and B. Further, prove that $A+B=\langle A \cup B \rangle$, the ideal generated by $A \cup B$. 1+4+5=10

Or

Show that the intersection of any family of subspaces of a vector space is again a subspace. Also show that union of two subspaces of a vector space is a subspace if and only if one is contained in the other. 5+5=10

(c) Let f be an endomorphism of the group G such that f commutes with every inner automorphism of G. Show that—

(i) $K=\{x \in G : f^2(x)=f(x)\}$ is a normal subgroup of G;

(ii) $\frac{G}{K}$ is Abelian. 5+5=10

Or

Let G be a finite group and p be a prime number such that $p / o(G)$. Prove that there exists $x \in G$ such that $o(x)=p$. 10

(d) If D is an ideal of a ring R, then show that there exists a one-one, onto mapping between the set of all ideals of R, containing D and the set of ideals of $\frac{R}{D}$. 10

Or

Show that any ring can be embedded into a ring with unity. 10

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