2016

MATHEMATICS

(Major)

Paper : 5.1

(Real and Complex Analysis)

Full Marks : 60

Time : 3 hours

The figures in the margin indicate full marks for the questions

1. Answer the following questions : $1 \times 7 = 7$

(a) State a sufficient condition for the continuity of a real-valued function of two variables.

(b) Give an example of a real-valued function which is bounded but not Riemann integrable.

(c) A real-valued function f is defined on $[a, b]$ having a singular point in its domain. State whether f is Riemann integrable or not.

(d) A function $f(z) = u(x, y) + iv(x, y)$ is defined such that

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$$

State whether f is analytic or not.

(e) Evaluate

$$\frac{1}{2\pi i} \oint_C \frac{\cos \pi z}{z^2 - 9} dz$$

where C is a closed rectangle with vertices at $z = 2 \pm i, -2 \pm i$.

(f) State Cauchy's integral formula.

(g) Define conformal mapping.

2. Answer the following questions : 2×4=8

(a) Discuss the continuity of the following function at (0, 0) :

$$f(x, y) = \frac{xy^3}{x^2 + y^6}, \quad (x, y) \neq (0, 0)$$
$$= 0, \quad (x, y) = (0, 0)$$

(b) Show that the integral

$$\int_0^1 x^{m-1} e^{-x} dx$$

is convergent for $m > 0$.

(c) Prove that if $w = f(z) = u + iv$ is analytic in a region R, then

$$\frac{dw}{dz} = \frac{\partial w}{\partial x} = -i \frac{\partial w}{\partial y}$$

where u, v are functions of two variables x, y.

(d) Find the fixed points of the transformation $w = \frac{2z - 5}{z + 4}$.

3. Answer any *three* parts : 5×3=15

(a) If

$$u = \cos x, \; v = \sin x \cos y$$
$$w = \sin x \sin y \cos z$$

then show that

$$\frac{\partial(u, v, w)}{\partial(x, y, z)} = (-i)^3 \sin^3 x \sin^2 y \sin z$$

The symbols have their usual meanings.

(b) Prove that if f is a bounded function on $[a, b]$, then to every $\varepsilon > 0$, there corresponds $\delta > 0$ such that

$$U(p, f) < \int_a^b f dx + \varepsilon$$

The symbols have their usual meanings.

(c) Show that the integral

$$\int_0^{\pi/2} \log \sin x \, dx$$

is convergent. Hence evaluate it.

(d) Prove that if $f(z)$ and $g(z)$ are analytic at z_0 and $f(z_0) = g(z_0) = 0$ but $g'(z_0) \neq 0$, then

$$\lim_{z \to z_0} \frac{f(z)}{g(z)} = \frac{f'(z_0)}{g'(z_0)}$$

(e) Prove that if $f(z)$ is integrable along a curve C having finite length L and if there exists a positive number M such that $|f(z)| \leq M$ on C, then

$$\left| \int_C f(z) \, dz \right| \leq ML$$

4. Answer *either (a) or (b)* :

(a) *(i)* If v is a function of two variables x and y, and $x = r\cos\theta$, $y = r\sin\theta$, then prove that

$$\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} = \frac{\partial^2 v}{\partial r^2} + \frac{1}{r^2}\frac{\partial^2 v}{\partial \theta^2} + \frac{1}{r}\frac{\partial v}{\partial r} \quad 5$$

(ii) Find the shortest distance from the origin to the hyperbola

$$x^2 + 8xy + 7y^2 = 225, \; z = 0 \quad 5$$

(b) *(i)* Verify the convergence of the integral

$$\int_0^{\infty} \frac{x \tan^{-1} x}{(1 + x^4)^{1/3}} \, dx \quad 5$$

(ii) Find the value of p such that

$$\int_1^{\infty} \frac{\sin x}{x^p} \, dx$$

converges absolutely. 5

5. Answer *either (a) or (b)* :

(a) *(i)* Prove that if a function f is Riemann integrable on $[a, b]$, then f^2 is also Riemann integrable on $[a, b]$. 5

(ii) A function f is defined on $[-1, 1]$ as follows :

$$f(x) = 1, \; x \neq 0$$
$$= 0, \; x = 0$$

Show that f is integrable on $[-1, 1]$ and calculate its value. 5

(b) *(i)* Show that the function f defined as follows

$$f(x) = \frac{1}{2^n}, \text{ when } \frac{1}{2^{n+1}} < x \le \frac{1}{2^n} \, (n = 0, 1, 2, \cdots)$$
$$= 0 \;, \; x = 0$$

is integrable on [0, 1]. Also evaluate

$$\int_0^1 f \, dx$$ 5

(ii) If f and g are both differentiable on $[a, b]$ and if f', g' are both integrable on $[a, b]$, then show that

$$\int_a^b f(x) \, g'(x) \, dx = [f(x) \, g(x)]_a^b - \int_a^b g(x) f'(x) \, dx$$ 5

6. Answer *either (a) or (b)* :

(a) *(i)* If

$$u_1(x, y) = \frac{\partial u}{\partial x}$$

and $u_2(x, y) = \dfrac{\partial u}{\partial y}$

then prove that

$$f'(z) = u_1(z, 0) - i u_2(z, 0)$$ 5

(ii) Prove that $\dfrac{d}{dz}(z^2 \, \bar{z})$ does not exist anywhere. 5

(b) *(i)* Evaluate

$$\oint_C \bar{z}^2 \, dz$$

around the circle $|z - 1| = 1$. 5

(ii) Find a bilinear transformation which maps $z = 0, -i, -1$ into $w = i, 1, 0$ respectively. 5

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